Bioengineering Science 2 - Formula Summary

1 Quantities, Units, and Dimensionless Numbers

Quantity	Unit	Quantity	Unit
Temperature, T	K	Length, L	m
Mass, m	kg	Time, t	s
Energy, E	J	Surface area, A	m^2
Density, ρ	kg/m^3	Specific heat capacity, c_p	J/kg K
Heat flow, Q	W or J/s	Heat flux, q	W/m^2
Thermal conductivity, k	W/(m K)	Convective coeff., h	W/(m^2 K)
Heat source per unit volume, $\dot{S}_v$	W/m^3	Thermal diffusivity, α	m^2/s
Thermal resistance, R_T	K/W	Time constant, τ	s
Effective momentum diffusivity, ν	m^2/s	Dynamic viscosity, μ	kg/(m s)
Concentration, C	Kmol/m^3	Diffusivity, $\mathcal{D}$	m^2/s
Convective mass transport coeff., K_m	m/s	Mass flux, $\mathbf{j}$	Kmol/m^{2}s

Reynolds number (Re_x), Prandtl number (Pr), Biot number (Bi) and Nusselt number (Nu) are dimensionless.

Reynolds Number	$Re_x = \frac{u_\infty x}{\nu}$	Prandtl Number	$Pr = \frac{\nu}{\alpha}$
Biot Number	$Bi = \frac{hL}{k}$ (k for solid)	Nusselt Number	$Nu = \frac{hL}{k}$ (k for fluid)
Pèclet Number	$Pe = \frac{UL}{\mathcal{D}_{AB}}$	Schmidt Number	$Sc = \frac{\nu}{\mathcal{D}}$
Sherwood Number	$Sh = \frac{k_m L}{\mathcal{D}}$		

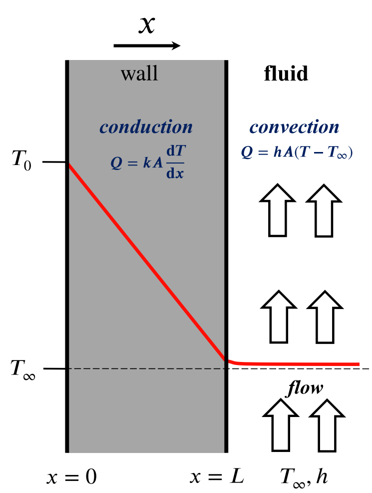
2 Heat Transport

- Conduction, Fourier's Law:

$$\dot{Q} = -kA\frac{\mathrm{d}T}{\mathrm{d}x}$$

- Convection, Newton's Law of cooling:

$$\dot{Q} = hA(T_s - T_\infty)$$



2.1 Reynolds Transport Theorem, Heat Equation

- Reynolds Transport Theorem:

$$\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{CV} \rho\beta \mathrm{d}V + \oint_{CS} \rho\beta(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A = \dot{Q} - \dot{W} + \dot{S}$$

- Integral form of heat equation:

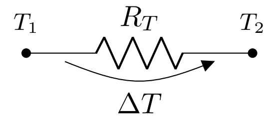
$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \rho c_p T \mathrm{d}V}_{\text{rate of change of thermal energy}} = \underbrace{\int_{CV} \dot{S}_v \mathrm{d}V}_{\text{rate of heat generation}} - \underbrace{\oint_{CS} (\mathbf{q}\cdot\mathbf{n})\mathrm{d}A}_{\text{rate of heat loss due to heat flux}} - \underbrace{\oint_{CS} \rho c_p T(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A}_{\text{rate of heat loss by fluid flow across CS}}$$

- Differential form of heat equation:

$$\underbrace{\rho c_p\left(\frac{\partial T}{\partial t}+(\mathbf{v}\cdot\nabla)T\right)}_{\text{rate of change of heat in fluid particle}} = \underbrace{\dot{S}_v}_{\substack{\text{rate of}\\ \text{heat generation}}} + \underbrace{k\nabla^2 T}_{\substack{\text{rate of heat}\\ \text{accumulation by conduction}}}$$

2.2 Steady-State Conduction

	Thermal Resistance, R_T(K/W)
Conduction(planar)	L/kA
Conduction(cylindrical)	$\dfrac{\ln(r_2/r_1)}{2\pi Lk}$
Conduction(spherical)	$\dfrac{1}{4\pi k}(\dfrac{1}{r_1}-\dfrac{1}{r_2})$
Convection	$1/hA$



$$\dot{Q}=\frac{\Delta T}{R_T}$$

2.3 Transient Heat Conduction

- Lumped capacitance method:

$$T=(T_i-T_\infty)\ \exp(-\frac{t}{\tau})+T_\infty \quad \Rightarrow \frac{T-T_\infty}{T_i-T_\infty}=\exp(-\frac{t}{\tau}),\quad \tau=\frac{\rho c_p V}{hA}$$

- Conduction through semi-infinite solid:
  - Penetration depth of the "front": $x\sim\sqrt{\alpha t}$
  - Error function: $1-\text{erf}(z)=\text{erfc}(z)$

$$T=(T_s-T_i)\ \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)+T_i$$

  - Heat flux at the surface:

$$q_s=\sqrt{\frac{k\rho c_p}{\pi t}}(T_s-T_i)$$

  - Interfacial temperature:

$$T_s=\frac{T_A m_A+T_B m_B}{m_A+m_B},\quad m_A=\sqrt{k_A\rho_A c_{p,A}},\quad m_B=\sqrt{k_B\rho_B c_{p,B}}$$

2.4 Convective Heat Transfer

- Averaged convective coefficient: $\bar{h}=\dfrac{1}{A}\displaystyle\int h\ \mathrm{d}A_S$
- Thickness of the velocity boundary layer: $\delta_v\sim\sqrt{\nu t}$, where $\nu=\mu/\rho$
- Thickness of the thermal boundary layer: $\delta_T\sim\sqrt{\alpha t}$
- Given a thermal boundary layer, to find h:

$$h=-\frac{k}{T_s-T_\infty}\frac{\mathrm{d}T}{\mathrm{d}y}\bigg|_{y=0}$$

- Reynolds Number

$$Re_x=\frac{\rho u_\infty x}{\mu}=\frac{u_\infty x}{\nu}$$

- Laminar flow over an isothermal *flat plate*: $Re_x << 5\times10^5$

$$Pr << 1 \quad Nu_x = 0.564 Re_x^{\frac{1}{2}} Pr^{\frac{1}{2}} \quad \bar{Nu}_L = 1.128 Re_L^{\frac{1}{2}} Pr^{\frac{1}{2}}$$

$$Pr > 0.5 \quad Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} \quad \bar{Nu}_L = 0.664 Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

- Turbulent flow over an isothermal *flat plate*: $5 \times 10^5 < Re_x < 3 \times 10^7$

$$0.7 < Pr < 400 \quad Nu_x = 0.029 Re_x^{0.8} Pr^{0.43}$$
$$\bar{Nu}_L = 0.664 Re_c^{\frac{1}{2}} Pr^{\frac{1}{3}} + 0.036 Re_L^{0.8} Pr^{0.43} \left[1 - \left(\frac{Re_c}{Re_L} \right)^{0.8} \right]$$

- Laminar, fully developed flow through a *circular* pipe: $Re_D << 2300$

Uniform surface heat flux (uniform q_h)	$Nu_D = 4.36$
Uniform surface temperature (uniform T_s)	$Nu_D = 3.66$

- Turbulent flow, fully developed flow through a *circular* pipe: $Re_D > 10^4$, $Pr > 0.7$

$$Nu_D = 0.023 Re_D^{0.8} Pr^{0.4}$$

- Internal flow:
  - Hydrodynamic entrance length: $x_{e,V} \approx 0.05 D\ Re_D$
  - Thermal entrance length: $x_{e,T} \approx 0.05 D\ Re_D\ Pr$
  - Mean temperature: $T_m = \dfrac{2\pi}{Q} \displaystyle\int_0^R T\ u\ r\ \mathrm{d}r$

3 Mass Transport

Advection	Diffusion	Convection
$\mathbf{j}_a = C\mathbf{u}$	$\mathbf{j}_d = -\mathcal{D}\nabla C$	$\mathbf{j}_k = k_m(C - C_m)$

$$\Rightarrow \quad \mathbf{j}_{\text{total}} = \mathbf{j}_a + \mathbf{j}_d = C\mathbf{u} - \mathcal{D}\nabla C$$

- Fick's First Law in 1-D

$$j = -\mathcal{D}\frac{\partial C}{\partial x}$$

- Stokes-Einstein Equation

$$\mathcal{D} = \frac{k_B T}{6\pi\mu a}, \quad a = \left(\frac{3M_w}{4\pi\rho N_A} \right)^{\frac{1}{3}}$$

- Integral form of conservation of solute of mass

$$\frac{\partial}{\partial t}\int_{CV} C \mathrm{d}V = \int_{CV} \dot{S}_v \mathrm{d}V - \oint_{CS} (\mathbf{j}\cdot\mathbf{n})\mathrm{d}A - \oint_{CS} C(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A$$

- Differential form of conservation of solute of mass

$$\frac{\partial C}{\partial t} + (\mathbf{v}\cdot\nabla)C = \mathcal{D}\nabla^2 C + \dot{S}_v$$